

# Improving Quantum State Estimation with Mutually Unbiased Bases

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When used in quantum state estimation, projections onto mutually unbiased bases have the ability to maximize information extraction per measurement and to minimize redundancy. We present the first experimental demonstration of quantum state tomography of two-qubit polarization states to take advantage of mutually unbiased bases. We demonstrate improved state estimation as compared to standard measurement strategies and discuss how this can be understood from the structure of the measurements we use. We experimentally compared our method to the standard state estimation method for three different states and observe that the infidelity was up to  $1.84 \pm 0.06$  times lower using our technique than it was using standard state estimation methods.

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Quantum state estimation is a central problem in the field of quantum information, with applications in quantum cryptography, quantum computing, quantum control, quantum measurement theory and foundational issues in quantum mechanics. The practical techniques used in quantum state estimation such as quantum state tomography[1] have been pivotal in the recent progress of experimental quantum mechanics. Many of the major advances in the field including demonstrations of entanglement of two[2], three[3], and four-photon states[4], quantum logic gate characterization[5], implementation of Shor's algorithm[6], and cluster state quantum computing[7] used quantum state tomography as the main diagnostic and descriptive tool. Quantum state tomography has by now been applied to nearly all candidate systems proposed for quantum information and computation including trapped ions[8], spontaneous parametric downconversion sources[1, 9, 10], atomic ensemble quantum memories[11], atoms trapped in optical lattices[12], cavity QED systems[13], quantum dot sources of entangled photons[14] and superconducting quantum bits[15]. Improved techniques for quantum state tomography therefore impact a wide range of applications in experimental physics. Moreover, because measurement plays such a central role in quantum mechanics, quantum state estimation provides one of the best conceptual tools we have for understanding what quantum states *are*[16, 17].

While all tomographic schemes that have been implemented to date for more than one particle use projection-valued measurements (PVMs), so far none has employed the optimal set of PVMs, the set with the property of being *mutually unbiased*. As we will show, mutual unbiasedness removes informational redundancy among different measurements, a limitation for all previous multi-particle state estimation strategies. We experimentally demonstrate the increase in measurement precision that can be achieved by employing these special sets of measurements.

Quantum state tomography on qubits involves the measurement of some linearly-independent,

informationally-complete or over-complete set of expectation values. A reconstruction based on linear inversion[18], maximum-likelihood fitting[1] or an appropriate cost function[19] is then used to calculate the best-fit density matrix for the data set.

All two-qubit quantum state tomography implementations to date have constructed a complete, linearly independent set of projectors from pairwise combinations of eigenstates of the Pauli operators[18]. Initial implementations employed 16 projectors[1], the minimum number required to satisfy linear-independence and completeness requirements. Later it was observed that an improved estimate of the density matrix could be obtained by performing tomography with projections onto all 36 tensor products of Pauli eigenstates[20, 21]. These 36 projectors can be arranged into nine bases of four orthogonal projectors as shown in the left column of Table I. We will refer to this tomography strategy as *standard separable quantum state tomography* (SSQST).

On the face of it, this set of 36 projectors appears unbiased. Certainly no particular basis or direction is preferred over any other. If, however, one looks at pairs of bases, then one notices that some bases share eigenstates of a particular Pauli operator while others have no eigenstates in common.

The overlap among projectors from different bases can be measured using the Hilbert-Schmidt overlap[18]. Projectors from the first and second bases of the left side of Table I that share an eigenstate for the first qubit have an overlap of 0.5 whereas those that are orthogonal in the first qubit have an overlap of zero. In contrast, bases that differ in Pauli operators for both qubits have an overlap of 0.25 for all pairs of projectors.

This inequivalence between pairs of bases constitutes a bias in the measurement scheme. This bias, which will occur for any complete set of separable projectors, creates redundancy and limits the efficiency with which new information about the state can be collected. This is because in schemes that only contain separable measurements, correlations can only be observed in one basis at a time. In contrast, schemes that employ joint or entan-

SSQST	MUB QST
$HH, HV, VH, VV$	$HH, HV, VH, VV$
$HD, HA, VD, VA$	$RD, RA, LD, LA$
$HR, HL, VR, VL$	$DR, DL, AR, AL$
$DH, DV, AH, AV$	$\frac{1}{\sqrt{2}}(RL + iLR), \frac{1}{\sqrt{2}}(RL - iLR),$
$DD, DA, AD, AA$	$\frac{1}{\sqrt{2}}(RR + iLL), \frac{1}{\sqrt{2}}(RR - iLL)$
$DR, DL, AR, AL$	$\frac{1}{\sqrt{2}}(RV + iLH), \frac{1}{\sqrt{2}}(RV - iLH),$
$RH, RV, LH, LV$	$\frac{1}{\sqrt{2}}(RH + iLV), \frac{1}{\sqrt{2}}(RH - iLV)$
$RD, RA, LD, LA$	
$RR, RL, LR, LL$	

TABLE I: The measurement bases used in standard separable quantum state tomography (SSQST) and mutually unbiased basis quantum state tomography (MUB QST).

gling measurements are capable of probing correlations in multiple single-qubit bases at once. For example, a singlet-state projection onto  $\frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$  simultaneously probes anti-correlation in all bases at once while a measurement of  $|HV\rangle$  only determines the degree of correlation in  $\sigma_z \otimes \sigma_z$ , but provides no information about correlation in  $\sigma_y \otimes \sigma_y$  or  $\sigma_x \otimes \sigma_x$ .

As a figure of merit to gauge the accuracy of an estimation technique we use the *infidelity*, which characterizes the distance between two density matrices  $\sigma$  and  $\rho$ . The infidelity is defined as  $1 - F$  where  $F$  is the fidelity[22],

$$F = \left( \text{Tr} \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}} \right)^2. \quad (1)$$

While there are other figures of merit that one could adopt, the fidelity has some appealing operational and analytic properties that make it well-suited to the task. A more detailed discussion of different figures of merit can be found in [21].

By this measure, SSQST will generally produce better estimates of separable states than of entangled states. This can be observed in the Monte-Carlo generated data in figure 1(a) where the infidelity estimate is plotted as a histogram over randomly selected maximally-entangled and separable states. The states were selected randomly over the Haar measure induced by local unitary transformations on the states[23] and the estimate was obtained by performing maximum-likelihood fitting on a simulated data set with on average 18,000 copies of the state. On average, the infidelity is significantly lower for separable states than for maximally-entangled states. The median values of the infidelity for separable and maximally-entangled states are  $0.0054 \pm 0.0001$  and  $0.0091 \pm 0.0002$  respectively.

If the class of measurement bases used in tomography is augmented to include entangled bases then this limitation of SSQST can be overcome. Indeed it is then possible to achieve *optimal* projective quantum state tomography, that is to say quantum state tomography with no informational redundancy. This can be achieved by taking advantage of mutually unbiased bases.

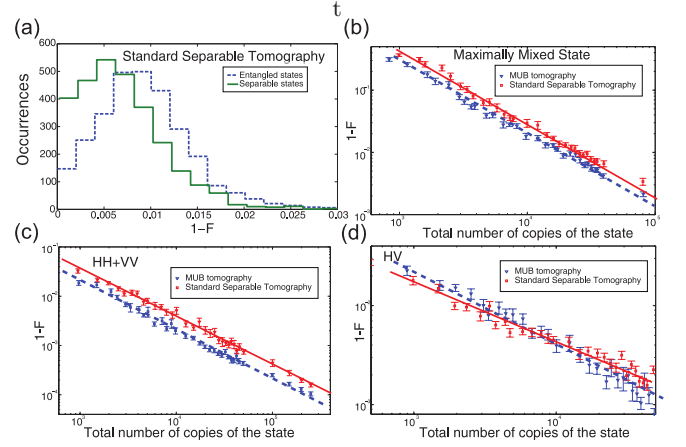


FIG. 1: (a) Histogram of infidelity for 3000 randomly selected entangled states and 3000 separable states for which SSQST was simulated. The Monte-Carlo simulation used 18,000 total copies for each random state. (b) Comparison of experimental MUB tomography and SSQST for the maximally mixed state. The average value of the ratio of infidelity using the SSQST-estimated density matrix to that using MUB tomography was  $1.49 \pm 0.05$ . (c) Comparison of experimental MUB tomography and SSQST for the state  $\frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$ . The average ratio of infidelity for the two methods was  $1.84 \pm 0.06$ . (d) Comparison of experimental MUB tomography and SSQST for  $|HV\rangle$ . The infidelity ratio for the two tomography methods was  $1.09 \pm 0.4$ . In (b) through (d) the solid line represents a fit to the SSQST data and the dotted line to the MUB QST data. The apparent crossing of the MUBs and SSQST lines in (d) is due to numerical errors at high  $N$ .

Mutually unbiased bases (MUBs), first introduced in the context of quantum state estimation by Wootters and Fields[24], have the property that all inner products between projectors of different bases labeled  $\alpha$  and  $\beta$  are equal. As a consequence,

$$\text{Tr}[P_{\alpha,\gamma} P_{\beta,\delta}] = 1/D, \quad (2)$$

whenever  $\alpha \neq \beta$ , where  $\gamma$  and  $\delta$  label the PVM elements for each basis and  $D$  is the dimensionality of the Hilbert space. The minimal number of MUBs needed for informational completeness is  $D + 1$  since each basis provides  $D - 1$  independent parameters plus a normalization, and  $(D - 1)(D + 1) = D^2 - 1$  is the number of free parameters in the density matrix. As it turns out,  $D + 1$  MUBs are always informationally complete when they exist and  $D + 1$  is the maximum number of MUBs that *can* exist. For qubits  $D = 2^N$ , and so the  $2^N - 1$  MUBs bases required for complete tomography is considerably smaller than the  $3^N$  Pauli bases used in conventional separable tomography. This can result in a considerable reduction in experiment time if significant resources are required to change measurement bases.

MUBs can be shown to exist whenever  $D$  is the power of a prime, as is the case for all multi-qubit systems[24]. The existence of a complete set of MUBs in cases where the dimensionality is not a power of a prime is an unre-

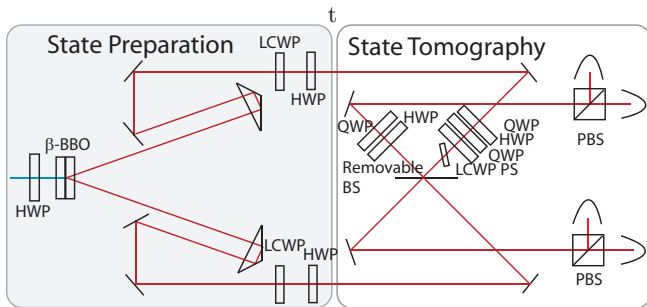


FIG. 2: Experimental apparatus for MUB state tomography. A non-linear crystal (BBO) is pumped to produce pairs of SPDC photons. Liquid crystal waveplates (LCWP), and half-waveplates (HWP) rotate the state. Entangling measurements are made using two-photon interference at a removable non-polarizing beamsplitter (BS), and these measurements can be rotated with HWPs, quarter waveplates (QWPs) and LCWPs to generate all necessary projections. For separable measurements the BS is removed and ordinary polarization analysis performed at the detectors using the polarizing beamsplitters (PBS), HWPs and QWPs.

solved problem, although it is widely believed that they do not generally exist[25].

The property stated in equation 2 can be thought of as expressing a complete lack of redundancy among measurements. After measuring the projections in one basis, the probability distribution of possible outcomes in the next basis is uniform. In other words, *nothing* is known about the outcomes of future measurements from previous ones. More formally, it has been shown that MUBs allow the maximum reduction in the Shannon entropy per measurement[24] averaged over all states.

This advantage of MUBs is almost universally, although unconsciously, applied in state estimation of one-qubit systems. For these systems, mutually unbiased bases consisting of the eigenstates of the  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  operators have been the standard choice for tomographic measurements since the very first studies in polarimetry[26].

For systems of qubits MUBs can be constructed as mutual eigenstates of Pauli operators following the approach of [27]. The particular set used in this work is shown in the right column of table I. Three of the two-qubit MUBs are separable and two of them are maximally entangled, making them amenable to standard linear-optics techniques for projective measurements[28].

To study the advantage of MUBs for state estimation, we repeated the measurements required for tomography over 3000 0.2-second intervals for each basis for both SSQST and MUB tomography. During each interval an average of 28 photon pairs per basis were detected. We added together randomly selected data sets from among these trials to obtain different numbers  $N_{\text{tot}}$  of total counts summed over all bases. At total numbers above approximately  $10^5$ , calculation of the infidelity was limited by computational errors and the infidelity failed

to continue its monotonic decrease. The plots in figure 1 were truncated at a point before this limit. At each value of  $N_{\text{tot}}$  we performed maximum-likelihood fitting to find the density matrix most likely to have generated the dataset, and used the infidelity measure to compare it to the density matrix fit of the entire dataset containing all counts. This process was repeated 30 times per point and the infidelity was averaged to produce the plots in figure 1. The error bars represent the measured standard deviation over the 30 trials.

The experimental apparatus used to perform state tomography both in the mutually unbiased bases and in the standard separable bases is shown in figure 2. We generate our two-photon states by spontaneous parametric downconversion (SPDC) in two  $\beta$ -BBO crystals cut for type-I phasematching at a  $3^\circ$  opening angle[29]. Because the crystals had their axes oriented at  $90^\circ$  to each other, the source could produce states with a degree of entanglement controlled by the pump polarization. The crystals were pumped by a 405-nm diode laser, generating broadband SPDC centered at 810 nm. The polarizations of the two downconverted photons could be controlled by liquid crystal waveplates (LCWPs) and half waveplates (HWPs). This control, taken together with the control on the pump polarization, allowed the system to produce a wide variety of entangled and unentangled pure states. Due to the multimodal collection system and the 0.5 nm bandwidth of our pump laser, the purity of maximally entangled states generated with the system was 0.9. By applying random phase shifts with the LCWPs, mixed states could also be generated[30] with fidelity  $> 0.98$ .

Both maximally entangling and separable measurements are required for MUB tomography. For the maximally entangling measurements, a polarization rotation followed by two-photon interference on a 50-50 beamsplitter[31] was used.

The visibility of the two-photon interference was measured to be 93%. Simulations demonstrated that this imperfect visibility increased infidelity for a given number of counts by between zero and 6%, depending on the state. This imperfect visibility was taken into account in constructing the operator basis used in the maximum-likelihood fitting algorithm. Instead of consisting of projectors onto pure states, the entangling measurements were modeled as rank-2 operators equal to a weighted sum of a projector onto  $|\psi^+\rangle$  and a projector onto  $|\psi^-\rangle$ . This change did not affect the informational completeness of the whole set of measurements, but meant that a greater number of counts needed to be obtained to correctly estimate the expectation values of operators that depended on these partially mixed projectors.

The beamsplitter was mounted on a scissor jack and so could be removed from the optical path without changing the alignment, allowing us to implement separable measurements as well as entangling measurements. Standard polarization analysis enabled us to collect measurements from the three separable MUBs and the nine separable bases in SSQST.

Figure 1 shows plots of the infidelity ( $1 - F$ ) against total number of counts  $N_{\text{tot}}$  for representative mixed, entangled and separable states. Based on previous analyses [21] we expect the infidelity to drop as  $1/\sqrt{N_{\text{tot}}}$  for pure states and  $1/N_{\text{tot}}$  for maximally mixed states. The state  $|HV\rangle$  and the maximally mixed state agree with this prediction, but the infidelity of the entangled state  $\frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$  drops more rapidly than predicted. This effect is not currently understood but we are investigating it in follow-up experiments. The apparent crossing of the MUBs and SSQST infidelity lines is believed to be due to numerical errors for larger  $N$ .

The mixed state is analytically and conceptually simple because it generates a uniform probability distribution in all bases. For this state the ratio of SSQST infidelity to MUB QST infidelity was  $1.49 \pm 0.05$ , independent of  $N_{\text{tot}}$ . This result is reasonably consistent with the infidelity estimated analytically from the covariance matrix calculated from a linear inversion formula[1]. This analysis predicted a value of 1.38, but required the approximation that the inversion was linear instead of maximum-likelihood and that the error in the total counts for each basis was uncorrelated to the individual counts for each measurement in the basis. The smaller number of bases in MUB tomography allows 9/5 more measurements to be made in each for the same number of total copies of the state. Although SSQST provides a more complete covering of the Hilbert space by including a greater number of bases, this does not make up for the greater number of copies that MUB tomography can distribute to each of its minimal number of bases.

It might also be expected that MUBs offer a better estimate of entangled states. When we measured the state  $\frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$  with the two techniques we found that the ratio of infidelity observed with SSQST to that with MUB tomography was  $1.84 \pm 0.06$ . In this case, in addition to the better statistics obtained by having fewer bases, the MUB tomography is able to estimate the

strength of correlations in different bases without having to collect redundant information about the single-qubit polarization.

The advantage of MUB tomography as compared to SSQST approaches insignificance when we look at separable states. For  $|HV\rangle$  we observe that the ratio of infidelity for SSQST as compared to MUB QST is  $1.09 \pm 0.05$ . However, even the fact that MUBs are *no worse* at estimating separable states is indicative of their superior capabilities since this is the class of pure states that the standard separable tomography estimates best.

We have demonstrated optimal projective quantum state tomography on a number of quantum states and compared it with standard separable-state tomography. While high-quality entangling measurements remain difficult to make in quantum optical systems, for systems where strong entangling interactions are available, such as trapped ion quantum computers, MUB tomography may already be a good choice to reduce the complexity and the duration of quantum state estimation[32]. MUBs are the natural choice of tomographic bases because of their ability to eliminate redundant measurements and to provide the best estimate of a quantum state from measurements on a discrete number of copies. Because they are based on PVMs, they can be implemented relatively easily in multi-particle systems. This is in contrast to other theoretically optimal measurements like SIC-POVMs[33] and to schemes that require joint measurements over multiple copies of the state[34].

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